# A NATURAL ORDER IN DYNAMICAL SYSTEMS BASED ON CONLEY-MARKOV MATRIX

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ABSTRACT. We introduce a new concept called natural order to study properties of dynamical systems, especially their invariant sets. The concept is based on the classical Conley index theory and transition probabilities between neighborhoods of different invariant sets when random perturbations are added to the dynamical systems. The transition probabilities can be determined by Fokker-Planck equations and their collection forms a matrix called Conley-Markov matrix. In the limiting case when the random perturbations reduce to zero, Conley-Markov matrix recovers the information given by the standard Conley connection matrix. Conley-Markov matrix also produces a total order among the neighborhoods of its invariant sets for a general dynamical system. In particular, it give the order among the local extremal points if the dynamical system is a gradient flow of an energy functional. Consequently, the natural order can be used to determine the global minima for gradient systems. More importantly, it gives a total order from the least to the most stable invariant sets for general dynamical systems, which can be viewed as a generalization of Morse decompositions for the system even it is not a gradient like flow. Some numerical examples are shown to illustrate the Conley-Markov matrix and its properties.

## 1. INTRODUCTION

In this paper, we study the ranking properties for invariant sets of dynamical systems. Let us start by looking at the following simple gradient system as an example,

$$\mathrm{d}x = -\nabla f(x)\mathrm{d}t, \quad x \in \mathbb{R}^n, \tag{1.1}$$

where $f(x)$ is a given energy function. A great attention has been given to its critial points $\{x_i\}_{i=1}^N$ which are invariant trajectories of the dynamical system. This is because they often correspond to the extremal states of the energy function $f$ which are of great interest in practice. In fact, finding (globally) minimal energy states has been considered as a fundamental problem and widely exist in physics, chemistry, biology, economy, engineering and many other disciplines. Despite of extensive literature on optimizations, searching for global optimizers still remain challenging in many applications, especially when the function $f$ has complicated energy landscape with multiple local minimizers.

A major hurdle in global optimizations is that one has to find many, if not all, extremal points. And then compare the values of the energy function at those points to determine which ones are the global minimizers. Thus, it is highly desirable to create a descending (or ascending, and we assume descending for simplicity in this paper) order among the invariant sets of the dynamical systems so that the global minimizers can be identified at the bottom of the list.

Although there has not been any report, to the best of our knowledge, on efficient numercial methods to generate such an order of invariant sets for dynamical systems in practice, some well known studies, such as Morse thoery and Conley index, have given partial answers theoretically.

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For example, for a gradient like system, there exists a natural Lyapunov function, which is its energy, strictly decreasing along non-constant orbits. Therefore it provides a partial order between invariant sets. For general systems, Conley's Morse decomposition theorem presents that any compact metric spaces in which the states of dynamical systems are defined can be decomposed into finite number of Morse sets connected by some non-constant orbits. Each invariant set belongs to a Morse set. Conley defines a Lyapunov function which takes different constant values on different Morse sets and is strictly decreasing along trajectories outside these Morse sets so as to produce an order among them. Conley also proposes connection matrix to detect the transitions between Morse sets. Readers are refered to [9, 13, 14] for detials.

On the other hand, we must note that the order in Morse decomposition depends on the choice of Lyapunov functions. It is possible that different partial orders may be produced if different Lyapunov functions are selected. In addition, the existing theoretical studies are based on topological concepts such as homology, and cannot be easily applied to design efficient numerical methods for global optimizations. Moreover, the corresponding theory is virtually non-exist for random dynamical systems, which are widely used in applications.

A less obvious, but more important motivation for us to conduct the current study is that many dynamical systems, such as the non-gradient like systems, do not have an associated energy function. Therefore, we cannot use classical energy decreasing concepts to explain properties of their connecting trajectories among the Morse decomposition sets. However, it is commonly observed that certain invariant sets such as stable critical points or limit circles in many non-gradient like systems are more attractive (or preferred) than other invariant sets. The transition trajectories among the invariant sets always going in certain directions. For instance, more trajectories go to a sink than to sources or saddle points. Hence, it is natural to ask which invariant set is more stable with respect to small perturbations.

In [5], we have reported our initial results on how to extend Conley index theory to random dynamical systems. The main idea is using transition probabilities, which is governed by Fokker-Planck equations [25], among the invariant sets in conjunction with the classical Conley connection theory to form a Conley-Markov matrix. The Conley-Markov matrix provides a combination of topological information and probabilistic information for the invariant sets.

This paper is a continuation of our effort along this direction. The main goal is to introduce a natural order based on the Conley-Markov matrix for general dynamical systems. This is inspired by a simple observation from the Gibbs' distribution that more trajectories asymptotically accumulate in the neighborhoods of global minima for gradient systems when noise is presented. In other words, the probability of trajectories going to a small neighborhood of a global minimum is in general larger than to other regions provided the noise is small and the time is large. Similarly, it is also observed that some invariant sets of a general dynamical system may be more "attractive" than others. More precisely, we use Fokker-Planck equations to calculate the transition probabilities between neighborhoods of different invariant sets, and to form the Conley-Markov matrix with entries being the accumulated transition probabilities in a given time from one neighborhood to another neighborhood. A total ranking, we call it *natural order*, among the invariant sets can be given according to the collective information, column sum, of the Conley-Markov matrix. In other words, at the bottom of the ranking list we have the invariant sets that the trajectories are more likely to stay nearby.

It can be shown that this natural order recovers the order given by the energy function for gradient systems provided the noise is small and time is large. For general systems, the ranking given by Conley-Markov matrix assigns a positive real number to each neighborhood. And this is different from the classical Conley index theory in which only integers are assigned. This allow us to distinguish similar invariant sets that produce same Conley indices. Another difference is that we use neighborhood concepts instead of the invariant sets themselves. This avoid many technical complications in the associated theory. In addition, the natural order can be defined

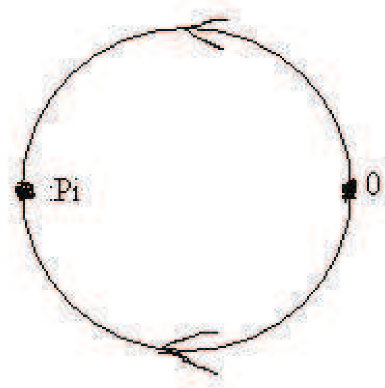


Figure 1. Phase portrait of $\dot{x} = \sin x$

for any finite time which is more computation friendly in practice. Moreover, one can view the ranking values as "energy" values for each invariant set, and therefore it is possible to apply many energy related techniques to study their properties.

The paper is organized as follows. In Section 2, we illustrate an example to compare some simple facts about selected known methods with the ranking given by Conley-Markov matrix. In Section 3, we review some relevant results in Conley index theory and random dynamical systems. In section 4, we introduce the Conley-Markov matrix, natural order for gradient systems. In Section 5, we obtain the similar results for general systems in finite dimensional spaces.

## 2. A simple example

In this section, let us use the gradient flow on the unit circle $S^1 = \mathbb{R}/(2\pi\mathbb{Z})$

$$\dot{x} = \sin x,$$

as an example to illustrate our purpose and method. As shown in Figure 1, it is easy to check that the invariant sets of the equation consist of two critical points: $\{0\}$ is a repeller and $\{\pi\}$ is an attractor.

Here, our goal is to study that:

(1) whether there are invariant sets near $\{0\}$ and $\{\pi\}$?
(2) if yes, whether there are connecting orbits between the invariant sets?
(3) which invariant set is the global minimum (or most stable state) of the system?

To answer these questions, let us analyze it by some existing methods.

2.1. **Classical Conley index method.** The initial step to find invariant subsets of the vector field for this problem can be the choice of upper semi-circle and lower semi-circle. By verifying the isolated invariant neighborhood, we have that $I(N) = \emptyset$. Then one can choose another left semi-circle and the right semi-circle.

(1) Let us consider the Conley index method (see [9, 10, 13, 14, 22, 23, 24]). Since we know that there maybe invariant sets near $\{0\}$ and $\{\pi\}$, we choose neighborhoods $\mathcal{N}_0$ and $\mathcal{N}_\pi$ of $\{0\}$ and $\{\pi\}$, respectively. Note that any trajectory starting from $\mathcal{N}_0$ will departure the neighborhood in some time, so the Conley index for the neighborhood $\mathcal{N}_0$ is a pointed

1-dimensional $[S^1]$. Therefore, by the Wazewski property of Conley index, we conclude that there is a nontrivial invariant set in $\mathcal{N}_0$. By computing $I(\mathcal{N}_0) = \{0\}$, we obtain that $\{0\}$ is actually an isolated invariant set. Similarly, if we consider the neighborhood $\mathcal{N}_\pi$, the Conley index for this neighborhood is a pointed 0-dimensional sphere $[S^0]$. Therefore, by contracting the neighborhood $\mathcal{N}_\pi$ again and again, we can conclude that $\{\pi\}$ is an isolated invariant set.

(2) Now we consider the connecting orbits between the two invariant sets $\{0\}$ and $\{\pi\}$. The two invariant subsets are the isolated critical points of a Morse function $\cos x$ for the gradient flow. Therefore, the connection matrix of Morse sets $\{0\}$ and $\{\pi\}$ is

$$\begin{pmatrix} 0 & \Delta(0,\pi) \\ 0 & 0 \end{pmatrix},$$

where $\Delta(0,\pi)$ is the connection mapping between the homology groups associated to $\{0\}$ and $\{\pi\}$. The homological connection $\Delta(0,\pi)$ counts the oriented number of trajectory flows from $\{0\}$ to $\{\pi\}$ with a fixed orientation. By taking the counterclockwise orientation, the trajectory from $\{0\}$ to $\{\pi\}$ on the lower semi-circle contributes $+1$, and the trajectory from $\{0\}$ to $\{\pi\}$ on the upper semi-circle contributes $-1$. Therefore $\Delta(0,\pi) = 0$. One can use this to obtain the homology information of the manifold $S^1$ as the following.

$$CH_*(\{S^1\}) = CH_*(\{0\}) \bigoplus CH_*(\{\pi\}).$$

Hence we can not conclude the existence of connecting orbits between $\{0\}$ and $\{\pi\}$ by the connection mapping.

(3) In this example, according to the Morse decomposition, the state space can be decomposed into two neighborhoods of the invariant sets and the connecting orbits between them, i.e. there is connecting orbit from $\{0\}$ to $\{\pi\}$. Therefore, $\{\pi\}$ should be the global minimum. But from the homological Conley connection matrix, there is no information about the existence of trajectory a prior. On the other hand, we can obtain a Conley-Markov matrix, which will be described later in this paper,

$$\begin{aligned} p_{12} &= p(0,\{0\};T_\varepsilon(\omega),\{\pi\}(T_\varepsilon(\omega))) = 0.95 \\ p_{11} &= p(0,\{0\};T_\varepsilon(\omega),\{\pi\}(T_\varepsilon(\omega))) = 0.05. \end{aligned}$$

$$P(T_\varepsilon(\omega)) = \begin{pmatrix} 0.05 & 0.95 \\ 0.95 & 0.05 \end{pmatrix}.$$

Now the matrix $P(T_\varepsilon(\omega))$ shows that there is a flow which goes from $\{0\}$ to $\{\pi\}(T_\varepsilon(\omega))$. This already finds the global maximum at $\{0\}$ and the global minimum at $\{\pi\}$ for the original problem.

2.2. **Method of random dynamical system.** We perturb the system via small white noise:

$$dx = \sin x dt + \epsilon dW(t), \tag{2.1}$$

where $W$ is the standard one-dimensional Brownian motion and the real line $\mathbb{R}$ is identified with $S^1 = \mathbb{R}/(2\pi\mathbb{Z})$. Under the white noise $\epsilon dW$, the trajectory starting from any point will travel everywhere of the whole circle almost surely. Therefore,

- there is no (random) invariant set in usual sense (in the random dynamical system), except for the trivial invariant sets $\emptyset$ and $S^1$;
- it is meaningless to consider connecting orbit in usual sense between invariant sets since any two sets can be connected.

The perturbed system generates a random dynamical system. Except for $\emptyset$, the only random invariant set is the whole circle $S^1$ since the additive white noise will push the particle traveling everywhere in a finite time no matter how small $\epsilon$ is. Therefore, even if let $\epsilon \to 0$, it seems that no information of (random) invariant sets, connecting orbits of the original unperturbed system can be obtained. Here, we would like to remark that the key in [5] is to use the white noise to obtain another vector field $\sin x dt + \epsilon dW(t,\omega)$. Then we can consider the usual dynamical system $dx(t,s) = (\sin x(t,s) + \epsilon W(t,\omega))ds$ for $(t,s) \in [0,T] \times \mathbb{R}$.

2.3. **Random Conley index method.** Note that the random Conley index obtained by Liu [19] for discrete time random dynamical system actually uses the random dynamical system method.

If we consider the time-1 map of the random dynamical system generated by (2.1), i.e. the time-discrete random dynamical system, we find that similar to the time continuous random dynamical systems, except for empty set, the only random invariant set with respect to the time-discrete random dynamical system is the whole circle, so the only random isolated invariant set is the whole circle and the corresponding random isolating neighborhood is also the whole circle. Therefore, we can not obtain any useful information about the original system by this method.

2.4. **Annealing method.** By classical annealing method, we can perturb the system as follows:

$$dx = \sin x dt + \sigma(t) dW(t),$$

where the $\sigma(t) = \frac{c}{\sqrt{\log t}}$ for large $c$, see [4] for details. The transition probability of the system will converge to the attractive critical point $\{\pi\}$ and the convergence time is *exponentially* long. At the same time, we note that the locally unstable rest point $\{0\}$ is invisible by this method. The critical point $\{0\}$ makes no difference to other points, except for $\{\pi\}$. Therefore, by annealing method, we can arrive at the global minimum of the original system (note that here the state space being compact is essential). The flaw is that this method ignores all the invariant sets that are not locally stable and that convergence time is too long.

2.5. **Fokker-Planck equation–a statistic method.** By considering the stochastic differential equation

$$dx = \sin x dt + \epsilon dW(t),$$

by the random dynamical system method in §2.2, there is no nontrivial invariant sets for the perturbed system no matter how small $\epsilon$ is. When $\epsilon$ is appropriately small, we only need to observe some time (not too long), we can find that the trajectories accumulate from $\{0\}$ invariant subset of the original vector field $X$ into the neighborhood of $\{\pi(t,\omega)\}$ of the invariant subset of the deformed vector field $X(t) = X - \epsilon W(t,\omega)$. Hence we can regain the same conclusion as that of annealing method, and the merit is that the time spent is relatively much smaller. Actually, this method can be used to study the connecting orbits between invariant sets and it is easy to use in numerical simulations (see also [5] for more discussions there).

## 3. Conley index theory and random dynamical systems

In this section, we briefly review the classical Conley index theory and related studies in random dynamical systems.

3.1. **Conley index theory.** In [14], Franzosa showed a refinement for Conley index pair and defined the Conley connection matrix for partially ordered Morse decompositions of isolated invariant sets. We briefly recall the concepts in this subsection.

Let $\Gamma$ be a Hausdorff topological space and the flow $\gamma \cdot t$ from $\Gamma \times \mathbb{R} \to \Gamma$ satisfies $\gamma \cdot 0 = \gamma, \gamma \cdot (t+s) = (\gamma \cdot t) \cdot s$ for every $\gamma \in \Gamma$ and $t, s \in \mathbb{R}$. A set $S$ is invariant if $S \cdot \mathbb{R} = S$. For any subset $U$, the $\omega$-limit set and the $\omega^*$-limit set of $U$ are given by $\omega(U) = \cap_{t>0} cl(U \cdot [t,\infty))$ and $\omega^*(U) = \cap_{t<0} cl(U \cdot (-\infty, t])$. Let $S \subset \Gamma$ be a compact invariant set and $U \subset S$. Both $\omega(U)$ and $\omega^*(U)$ are compact invariant subsets of $U$.

For two disjoint invariant sets $S_\pm$, the set of connections from $S_-$ to $S_+$ is defined by $C(S_-, S_+) = \{\gamma : \omega(\{\gamma\}) \subset S_+,\ \omega^*(\gamma) \subset S_-\}$, where $\omega, \omega^*$ are the omega and alpha limit sets of $\gamma$, respectively. Assume that the flow $\gamma$ is defined on a compact set $S$ ($S$ could be a compact invariant set in some larger space which need not be compact). A compact invariant subset $A$ of $S$ is called an *attractor* if it is the omega-limit set of some neighborhood of itself. Similarly, a compact invariant subset $R$ of $S$ is called a *repeller* if it is the alpha-limit set of some neighborhood of itself. Given an attractor $A$ and a repeller $R$, if for any $x \in X\backslash(A \cup R)$, the omega-limit of $x$ belongs to $A$ and the alpha-limit set of $S$ belongs to $R$, then $(A, R)$ is called an *attractor-repeller pair decomposition of* $S$ and $R$ is the repeller corresponding to $A$ (conversely $A$ is the attractor corresponding to $R$). The invariant set $S$ can be written $S = A \cup R \cup C(R, A)$, where For each attractor-repeller pair $(A, R)$, there is a Lyapunov function $L : S \to [0, 1]$ satisfying that $L$ takes on value 0 on $A$, takes on value 1 on $R$ and strictly decreases along the orbits outside of $A \cup R$.

A partial order set $(P, <)$ consists of a finite set $P$ along with a strict partial order relation $<$ with transitivity property only. An interval $I \subset P$ is a subset of $P$ such that given $i, j \in I$ and $i < k < j$ then $k \in I$.

**Definition 3.1.** For a flow $\gamma$ on a Hausdorff space $\Gamma$ and a compact invariant set $S \subset \Gamma$, a finite collection $\{M(\pi)|\pi \in P\}$ of compact invariant sets in $S$ is called a Morse decomposition of $S$ if there exists an ordering $\pi_1, \pi_2, \cdots, \pi_n$ of $P$ such that for every $\gamma \in S \setminus \cup_{\pi \in P} M(\pi)$ there exist indices $i, j \in \{1, 2, \cdots, n\}$ such that $i < j$ and $\omega(\gamma) \subset M(\pi_i)$ and $\omega^*(\gamma) \subset M(\pi_j)$. The ordering with the described property is called admissible, and the sets $M(\pi)$ are called Morse sets.

For the given Morse decomposition $\{M(\pi)|\pi \in P\}$ of compact invariant sets in $S$, there is a filtration of attractors $A_i, i = 1, \ldots, n$, and the associated filtration of repellers $R_i, i = 1, \ldots, n$ such that $(A_i, R_i), i = 1, \cdots, N$ are attractor-repeller pair decompositions of $S$ with

$$\emptyset = A_0 \subsetneq A_1 \subsetneq \cdots \subsetneq A_N = S \text{ and } S = R_0 \supsetneq R_1 \supsetneq \cdots \supsetneq R_N = \emptyset.$$

and that the $N$ sets given by

$$M_i = A_i \cap R_{i-1},\ 1 \le i \le N,$$

are just the Morse sets in the given Morse decomposition $\{M(\pi)|\pi \in P\}$. For any given Morse decomposition, there is a Lyapunov function $L : S \to [0, 1]$ satisfying that $L$ takes on different constant values on different Morse sets and $L$ is strictly decreasing along the orbits outside of Morse sets.

**Theorem 1** (Morse Decomposition Theorem [9])**.** *Any flow restricted to an invariant compact set decomposes the compact set into finite number of invariant compact subsets (i.e. Morse sets) and connecting orbits between them. There is a Lyapunov function which takes on different constant values on each Morse set and is strictly decreasing along orbits outside of Morse sets.*

In general, a Morse decomposition of a given invariant set $S$ with a prior partial ordering $P$ is a collection of finite compact invariant subsets of $S$ (invariant subsets are generalized critical points), and there are connecting orbits between these 'generalized' critical points according to the partial ordering. The global minimal point will be located in the generalized critical points such that the connection orbits flow in.

If $S$ is a compact invariant set and $\{M(\pi)|\pi \in P\}$ is a Morse decomposition of $S$, then for $\pi, \pi^{'} \in P$ one has the definition $\pi < \pi^{'}$ if $\pi \neq \pi^{'}$ and $\pi$ arrives before $\pi^{'}$ for every admissible ordering of $P$. This defines a partial order on $P$. A subset $I \subset P$ is an interval if $\pi^{'}, \pi^{''} \in I$ and $\pi \in P$ with $\pi^{'} < \pi < \pi^{''}$ then $\pi \in I$. We define $M(I) = (\cup_{i \in I} M(\pi_i)) \cup (\cup_{i,j \in I} C(M(\pi_i), M(\pi_j)))$ is also an isolated invariant set.

A compact invariant set $S$ is isolated if there exists a compact neighborhood $N$ of $S$ such that $S = I(N) := \{\gamma \in \Gamma | \gamma \cdot \mathbb{R} \subset N\}$. The compact subset $N$ is called an isolating neighborhood of $S$. An index pair $(N_1, N_0)$ for an isolated invariant set $S$ is a pair of compact sets in $\Gamma$ with properties (1) $N_0 \subset N_1$ and $N_1 \setminus N_0$ is a neighborhood of $S = I(cl(N_1 \setminus N_0))$ (2) $N_0$ is positively invariant in $N_1$ and (3) for $\gamma \cdot [0, \infty) \not\subset N_1$ there exists a $t \ge 0$ with $\gamma \cdot [0, t] \subset N_1$ and $\gamma \cdot t \in N_0$.

The concept of an index pair and the existence of index pairs plays a fundamental role in the Conley index theory for isolated invariant set (see [9, 10, 22]).

For admissible ordering of $M = \{M(\pi)\}_{\pi\in P}$, an index filtration $\{N(I)\}_{I\in A(<)}$ is a generalization of index pair established by Franzosa in [13, Definition 3.4]. If $(I,J) \in I_2(<)$, then $(M(I), M(J))$ is an attractor-repeller pair in $M(IJ)$. In particular, $M(I)$ is an attractor in $S$ with complementary repeller $M(P\setminus I)$ provided $I$ is an attracting interval in $(P,<)$. There is a family of compact forward invariant sets $\{N(I) : \text{for attracting intervals in } P\}$ such that

(i) $(N(J), N(I))$ is an index pair for $M(J\setminus I)$ for all attracting intervals $I\subset J$,

(ii) $N(I\cap J) = N(I)\cap N(J)$, $N(I\cup J) = N(I)\cup N(J)$ for all attractor intervals $I, J$.

Recall that $h(S)$ is the Conley index of $S$ as a homotopy type of a pointed topological space, and $CH_*(S;\mathbb{R}) = H_*(h(S);\mathbb{R})$ is the homological version of Conley index as a more computable object than the homotopy index. For compact closed oriented manifolds, the coefficients $\mathbb{R}$ can be replaced by the integer coefficients $\mathbb{Z}$. Let

$$\Delta : \bigoplus_{i\in P} CH_*(M(i);\mathbb{R}) \to \bigoplus_{i\in P} CH_*(M(i);\mathbb{R})$$

be a linear map such that $\Delta = (\Delta(i,j))_{i,j\in P}$ as a matrix map, where $\Delta(i,j) : CH_*(M(i);\mathbb{R}) \to CH_*(M(j);\mathbb{R})$. Similarly, we have $\Delta(I) = (\Delta(i,j))_{i,j\in I}$.

**Definition 3.2.** For a Morse decomposition $M(S)$ of an invariant set $S$, $\Delta$ is called a Conley connection matrix if (i) $\Delta$ is a upper triangular matrix, (ii) $\Delta\circ\Delta = 0$ if $\Delta(i,j)$ is of degree $-1$ for every $i,j$.

Note that the original Conley connection matrix definition in [13, 14] requires an isomorphism for each interval $I \in P$ between the homology of $\Delta(I)$ and $CH_*(M(I)$, and this isomorphism is compatible with long exact sequences induced by all pairs $(I,J)$ (see [14, 22]).

**Theorem 2.** *(1) There exists a Conley connection matrix for a Morse decomposition.*

*(2) The non-trivial connection entry $\Delta(i,j) \neq 0$ implies that $C(M(\pi_i), M(\pi_j)) \neq \emptyset$.*

*(3) Suppose that a Morse-Smale flow has no periodic orbits. Each nonzero map in the connection matrix is flow defined. The Conley connection matrix is unique.*

First two results are proved by Franzosa [13, 14], and the third by Reineck [23, 24]. Franzosa [14] constructed a non-unique Conley connection matrix at the bifurcation point.

*Without the partial ordering*, one can also define the Conley connection matrix for the general dynamical system with finitely many compact invariant subsets and the connecting orbits between the invariant subsets.

**Definition 3.3.** *The Conley connection matrix for the finitely many invariant subset $\{S_\alpha\}_{\alpha\in A}$ is given by*

$$\Delta_A : \bigoplus_{\alpha\in A} CH_*(S_\alpha;\mathbb{R}) \to \bigoplus_{\alpha\in A} CH_*(S_\alpha;\mathbb{R})$$

*a linear map such that $\Delta_A = (\Delta(\alpha,\beta))_{\alpha,\beta\in A}$, where*

$$\Delta(\alpha,\beta) : CH_*(S_\alpha;\mathbb{R}) \to CH_*(S_\beta;\mathbb{R}).$$

**Remarks:** (1) Although this is similar to the connection matrix defined in Franzosa [13, 14], our defined connection matrix $\Delta_A$ is not upper triangular since $A$ does not have any partial ordering.

(2) In fact we allow $\Delta(\alpha,\alpha)$ as the possible nonzero diagonals in the connection matrix. This definition is used in our previous result [5] for the invariant subsets of $X(t,\omega) = X - \varepsilon\xi(t,\omega)$ which is depending on the path of diffusion process $\xi(t,\omega)$ for finitely many invariant subsets in a compact manifold $M$.

3.2. **Random dynamical systems.** We give a brief review on random dynamic systems and introduce some notations are introduced in this subsection. See [1] for more details on this subject.

**Definition 3.4.** A measurable ($C^k, k = 0, \cdots, \infty, C^\omega$) random dynamical system on the measurable ($C^k, k = 0, \cdots, \infty, C^\omega$) space ($C^k, k = 0, \cdots, \infty, C^\omega$-manifold) $(X, \mathcal{B})$ over a metric dynamic system $(\Omega, \mathcal{F}, \mathbb{P}, (\theta(t))_{t\in\mathbb{T}})$ with time $\mathbb{T}$ is a mapping

$$\phi : \mathbb{T} \times \Omega \times X \to X, \quad (t, \omega, x) \mapsto \phi(t, \omega, x),$$

with the following properties:

(i) Measurability: $\phi$ is $(\mathcal{B}(\mathbb{T}) \otimes \mathcal{F} \otimes \mathcal{B}, \mathcal{B})$-measurable.

(ii) Regularity: $\phi(t, \omega) : X \to X$ is measurable ($C^k, k = 0, \cdots, \infty, C^\omega$).

(iii) Cocycle Condition: $\phi(0, \omega) = id_X$ for all $\omega \in \Omega$ and $\phi(t + s, \omega) = \phi(t, \theta(s)\omega) \circ \phi(s, \omega)$ for all $s, t \in \mathbb{T}, \omega \in \Omega$.

Given a random dynamic system, one can define the measurable skew product flow

$$\Theta(t)(\omega, x) := (\theta(t)\omega, \phi(t, \omega)x),$$

for all $t \in \mathbb{T}$. Here the skew product of the metric dynamic system $(\Omega, \mathcal{F}, \mathbb{P}, (\theta(t))_{t\in\mathbb{T}})$ and the cocycle $\phi(t, \omega)$ on $X$ gives a measurable dynamic system $\Theta(t) : (\Omega, \mathcal{F}, \mathbb{P}, (\theta(t))_{t\in\mathbb{T}}) \times X \to (\Omega, \mathcal{F}, \mathbb{P}, (\theta(t))_{t\in\mathbb{T}}) \times X$, and every such a measurable skew product dynamic system $\Theta$ defines a cocycle $\phi$ over a metric dynamic system $(\Omega, \mathcal{F}, \mathbb{P}, (\theta(t))_{t\in\mathbb{T}})$.

Suppose that the probability $\mu$ on $(\Omega \times X, \mathcal{F} \otimes \mathcal{B})$ is invariant for the skew product $\Theta$ (with respect to $\phi$), i.e. $\Theta(t)\mu = \mu$ for all $t \in \mathbb{T}$. A probability $\mu$ is an invariant measure for the random dynamic system $\phi$ if (i) $\mu$ is invariant for the skew product $\Theta$ and $\pi_\Omega\mu = \mathbb{P}$, where $\pi_\Omega : \Omega \times X \to \Omega$ is the canonical projection. By Theorem 1.2.10 of [1], the invariant measure for $C^0$ random dynamic system $\phi$ is non-empty, provided $X$ is a compact metric space. Let $\mathcal{P}(X)$ be the subsets of $X$, and $A : \Omega \to \mathcal{P}(X)$ be a function with values in $\mathcal{P}(X)$ (called a random set). Let $d_X$ be a metric on $X$.

**Definition 3.5.** (1) A map $A : \Omega \to \mathcal{P}(X)$ is called a random closed (compact) set if $\omega \mapsto d_X(x, A(\omega))$ is measurable for each $x \in X$ and $A(\omega)$ is closed (compact) for each $\omega \in \Omega$, where $d_X(x, A(\omega)) = \inf\{d_X(x, y) : y \in A(\omega) \subset X\}$.

(2) Let $C \subset \Omega \times X$. A random set $C$ is called forward (backward) invariant if $\phi(t, \omega)C(\omega) \subset C(\theta(t)\omega)$ $(C(\theta(t)\omega) \subset \phi(t, \omega)C(\omega))$ $\mathbb{P}$-a.s. for all $t \in \mathbb{T}_+$.

(3) A random set $C$ is an invariant if $\phi(t, \omega)C(\omega) = C(\theta(t)\omega)$ $\mathbb{P}$-a.s. for all $t \in \mathbb{T}$.

Note that Definition 3.5 is in Chapter 1 of [1] and an invariant set in Definition 3.5 is called strictly invariant in [1] . We can simply denote $C = \Theta(t)C$ for all $t \in \mathbb{T}$. Recently, Liu [19] has shown that there is a well-defined Conley index for discrete random dynamic systems through the Frank and Richeson's construction for maps. Extending the method in [19] to the flow case is still a challenging problem.

## 4. Conley-Markov matrix for gradient systems $\mathbb{R}^n$

We study the Conley-Markov matrix for the stochastic differential equation which is related to our original dynamical system.

Instead of the compact condition on manifolds in [5], we consider the flow generated by $V(x)$ in $\mathbb{R}^n$:

$$\begin{cases} \mathrm{d}x = V(x)\mathrm{d}t, \quad x \in \mathbb{R}^n \\ x(0) = x_0. \end{cases} \tag{4.1}$$

Adding a white noise on the dynamical system, we have the randomly perturbed the stochastic differential equation (or an Ito process)

$$\begin{cases} \mathrm{d}x = V(x)\mathrm{d}t + \sqrt{2\epsilon}\,\mathrm{d}W, \quad x \in \mathbb{R}^n, \\ x(0) = x_0, \end{cases} \tag{4.2}$$

where $W$ is an $n$-dimensional Brownian motion and $\epsilon$ is a positive constant.

We note that the solution of (4.2) is a Markov process on $[0, T]$. Associated to this Markov process, there corresponds a transition probability function $P(s, x; t, B)$[1], and this transition probability admits a density function, called *transition density function*, $p(s, x; t, y)$, such that

$$P(s, x; t, B) = \int_B p(s, x; t, y)\mathrm{d}y$$

for any Borel set $B \subset \mathbb{R}^n$. If the transition density function $p(s, x; t, y)$ is measurable in all its arguments, then one has the Kolmogorov–Chapman equation for the transition density function

$$p(s, x; t, y) = \int_{\mathbb{R}^n} p(s, x; u, z)p(u, z; t, y)\mathrm{d}z, \quad \forall\ 0 \le s \le u \le t.$$

See Chapter 3 of [16]. By the meaning of transition probability function $P(s, x; t, B)$, the Kolmogorov–Chapman equation for the transition probability follows

$$P(s, x; t, B) = \int_{\mathbb{R}^n} P(u, y; t, B)P(s, x; u, \mathrm{d}y), \quad \forall\ 0 \le s \le u \le t.$$

Note that, in (4.2), the coefficients are independent of $t$, so the corresponding Markov process is time homogeneous. In this case, we can simply denote $P(s, x; t, B)$ by $P(t - s, x, B)$ and $p(s, x; t, y)$ by $p(t - s, x, y)$. The transition density function $p(t, \xi, x)$ is a fundamental solution of the Fokker–Planck equation when it is regarded as a function of $(t, x)$:

$$\frac{\partial p}{\partial t} = \epsilon \Delta p - \nabla \cdot (pV), \quad (t, x) \in (0, +\infty) \times \mathbb{R}^n.$$

Let us consider the Cauchy problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = \epsilon \Delta u - \nabla \cdot (uV), \quad (x, t) \in \mathbb{R}^n \times (0, +\infty) \\ u(x, 0) = f(x), \quad x \in \mathbb{R}^n, \end{cases} \tag{4.3}$$

where $f(x) \ge 0$ is bounded, continuous or measurable, and satisfies $\int_{\mathbb{R}^n} f(x)\mathrm{d}x = 1$. If $V$ and $\mathrm{div}V$ are locally Hölder continuous and uniformly bounded and $f(x)$ is twice continuously differentiable, then nonnegative solution of (4.3) is unique by results in [16, Chapter 2, §9]. By classical parabolic theory, the solution $u$ can be explicitly expressed by

$$u(x, t) = \int_{\mathbb{R}^n} p(t, \xi, x)f(\xi)\mathrm{d}\xi. \tag{4.4}$$

This unique nonnegative solution $u(\cdot, t)$ of (4.3) is actually the density function of the Markov process associated to (4.2) at time $t$. In (4.2), the initial value $x_0$ is a random variable with density function given by $u_0$[2], then

$$\mathbb{P}(x(t) \in B) = \int_B u(t, x)\mathrm{d}x, \quad \forall\ B \subset B,$$

where $x(t)$ is the solution to the stochastic differential equation (4.2). When $\epsilon = 0$ in (4.2), the stochastic differential equation reduces to the ordinary differential equation (4.1). In this case,

[1]The transition probability function $P(s, x; t, B)$ means the probability that the stochastic trajectory entering the set $B$ at time $t$ if we start from $x$ at time $s$, here $0 \le s \le t$.

[2]This means that $\mathbb{P}(x_0 \in B) = \int_B u_0(x)\mathrm{d}x$ for any Borel set $B \subset \mathbb{R}^n$.

the initial point $x_0$ is a deterministic point, and the solution $x(t)$ is an ordinary solution of (4.1) such that

$$P(0,x;t,B) = \begin{cases} 1, & x(t) \in B, \\ 0, & x(t) \notin B. \end{cases}$$

For details on relations between Markov processes and Fokker-Planck equation, see [12, 16].

Consider the gradient system

$$\mathrm{d}x = -\nabla f(x)\mathrm{d}t, \quad x \in \mathbb{R}^n, \tag{4.5}$$

with the following assumptions.

(H1) $f$ has a global minimum and $\lim_{|x|\to\infty} f(x) = \infty$.

(H2) The system (4.5) has finite fixed points and all of them are isolated.

To study the gradient system (4.5), we consider the randomly perturbed system

$$\mathrm{d}x = -\nabla f(x)\mathrm{d}t + \sqrt{2\epsilon}\ \mathrm{d}W, \quad x \in \mathbb{R}^n. \tag{4.6}$$

Then the transition probability density function $u(x,t)$ of the perturbed system (4.6) satisfies the the Fokker–Planck equation:

$$u_t = \epsilon\Delta u + \nabla \cdot (u\nabla f). \tag{4.7}$$

Assume that $u^\epsilon$ is the stationary solution of the Fokker–Planck equation, i.e., $u^\epsilon$ satisfies the corresponding elliptic equation

$$\epsilon\Delta u^\epsilon + \nabla \cdot (u^\epsilon \nabla f) = 0.$$

We have that $u^\epsilon(x) = k\mathrm{e}^{-f(x)/\epsilon}$ is the stationary solution of (4.7) provided that

$$\int_{\mathbb{R}^n} \mathrm{e}^{-f(x)/\epsilon}\mathrm{d}x < \infty \tag{4.8}$$

and the constant $k^{-1} = \int_{\mathbb{R}^n} \mathrm{e}^{-f(x)/\epsilon}\mathrm{d}x$. The stationary solution $u^\epsilon$ is called *Gibbs measure* in statistical mechanics. One may add dissipation condition to guarantee (4.8) to ensure the existence of the Gibbs measure. For instance, if there exists constants $a, b > 0$ such that $f(x) > a|x|$ whenever $|x| > b$, which is a sufficient condition for (4.8). If $u(t,x)$ is the solution to (4.7) with initial condition $u(0,x) = u_0(x)$ where $u_0(x)$ satisfies $\int_{\mathbb{R}^n} u_0(x)\mathrm{d}x = 1$, under some mild conditions in [21, 25]), then one has

$$\int_{\mathbb{R}^n} u(t,x)\mathrm{d}x = 1, \text{ for all } t \geq 0 \quad \text{and} \quad \lim_{t\to\infty} u(t,x) = u^\epsilon(x).$$

Assume that $\{x_i \in \mathbb{R}^n : i = 1, \ldots, N\}$ is the set of isolated fixed points of the unperturbed system (4.5). Let

$$\delta = \min_{1\leq i,j\leq N} \|x_i - x_j\|.$$

We fix a time $T > 0$. By Lemma 3.5 of [5], there exists a time $T(\omega)$ such that the Conley index pair of $x_i$ is homotopic to a Conley index pair of $x_i(T(\omega)$, where $x_i(T(\omega)$ is the corresponding isolated fixed point of the dynamical system

$$\frac{dx}{ds} = -\nabla f(x(s)) + \sqrt{2\epsilon} dW(T(\omega), \omega),$$

for $s \in (-\infty, \infty)$. This is a stochastically deformed vector field from the original gradient system. Therefore we have our Conley transition matrix $\Delta_N[0, T(\omega)]$ can be studied by results from [9, 10, 22, 23]. In this paper, we focus on our Conley-Markov matrix defined in [5] with time average.

Assume that nonnegative continuous functions $u_0^i$, $i = 1, \ldots, N$, satisfy $u_0^i(0) = x_i, u_0^i(T) = x_j(T)$ and that $\int_{\mathbb{R}^n} u_0^i(x)\mathrm{d}x = 1$. Consider the Cauchy problem

$$\left\{ \begin{array}{lcl} u_t & = & \epsilon\Delta u + \nabla \cdot (u\nabla f), \quad x \in \mathbb{R}^n \\ x_i & = & u_0^i(x) \end{array} \right. \tag{4.9}$$

with $u(0, x) = u_0^i$ and assume that $u^i(t, x)$ is the solution to (4.9).

**Definition 4.1.** Let

$$m_{ij}^\epsilon := \frac{1}{T}\int_0^T u^i(t, x)\mathrm{d}x\mathrm{d}t,$$

which stands for the average probability that the trajectory of (4.6) starting from $B(x_i, \delta_i)$ to $B(x_j, \delta_j)$ over the time period $T$. The *Conley-Markov matrix wtih $\epsilon$-diffusion $M^\epsilon(T)$* of (4.5) is the $N \times N$ matrix with entries given by $m_{ij}^\epsilon$:

$$M^\epsilon(T) = \begin{pmatrix} m_{11}^\epsilon & m_{12}^\epsilon & \cdots & m_{1N}^\epsilon \\ m_{21}^\epsilon & m_{22}^\epsilon & \cdots & m_{2N}^\epsilon \\ \vdots & \vdots & \ddots & \vdots \\ m_{N1}^\epsilon & m_{N2}^\epsilon & \cdots & m_{NN}^\epsilon \end{pmatrix}.$$

The elements of the Conley-Markov matrix are probabilities values related to time $T$, the perturbation parameter $\epsilon$, and the neighborhood $B_j$. For convenience, we also use notations $M$ or $M^\epsilon$ if they do not cause any confusion.

**Definition 4.2.** Let $S_j^\epsilon(T) = \sum_{i=1}^N m_{ij}^\epsilon$, the sum of the $j$-th column of $M^\epsilon(T)$, $j = 1, \ldots, N$. $S_j$ is called the *natural energy of $x_j$*.

The natural energy of $x_j$ denotes the total 'probability' that the trajectory starting from all the fixed points $x_i, i = 1, \ldots, n$ will connect to $x_j$ in the period $[0, T]$ through the stoachastic process (4.6). Note that $S_j^\epsilon$ may be greater than 1 since $0 \le m_{ij} \le 1$ and hence $0 \le S_j^\epsilon \le N$. Thus it is reasonable to believe that the the global minimal points are actually $x_k$ with $k$ satisfying

$$S_k^\epsilon = \max_j S_j^\epsilon.$$

**Definition 4.3.** The *natural order '$\leqq_\epsilon$'* among $\{x_j; j = 1, \ldots, N\}$ is defined by

- $x_i <_\epsilon x_j$ if and only if $S_i^\epsilon(T) < S_j^\epsilon(T)$;
- $x_i =_\epsilon x_j$ if $S_i^\epsilon(T) = S_j^\epsilon(T)$.

From the definition, it is clear that the natural order among $\{x_j; j = 1, \ldots, N\}$ depends on the choice of $\epsilon$, $T$ and the neighborhood selection $B_j$. This seems to be in favorable as the order may change if different parameters are chosen. However, we would like to use one simple example to argue that this flexible property may be desirable in many practical problems, especially when finite time behavior of the invariant sets is the subject of study. In addition, with proper selections of the parameters, the natural order can recover the order induced by the original energy function $f$.

Let us consider the following system with the energy function $f$ given in Figure 2. There are five critical points in the gradient system with three stable sinks and two unstable source. The middle one $x_3$ is the global minimizer for the function $f$. However, if the initial state is uniformly distributed in the region and noise level $\epsilon$ is small, the trajectories will have much larger probability staying near the local minima $x_1$ and $x_5$ in a finite time interval $[0, T]$ than staying in the neighborhood of $x_3$. Therefore, it is more likely to observe $x_1$ or $x_5$ than $x_3$. In fact, $x_3$ is only more attractive when $T$ is large enough.

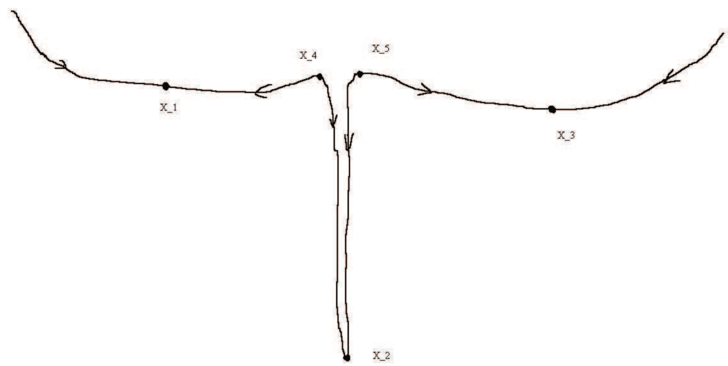


FIGURE 2. The energy function $f$ with 5 critical points. Although the middle one is the global minima, it is less likely to be observed in a finite time interval if the initial state is uniformly distributed in the region.

The dependence of the natural order on the selection of the neighborhood $B_j$ become relative less crucial if noise is weak enough and time $T$ is large enough. This can be seen in the following theorem.

**Theorem 3.** *For any* $0 < \eta$, *there exists* $\epsilon_0(\eta) > 0$ *and* $T_0(\eta) > 0$ *such that for* $\epsilon < \epsilon_0(\eta)$ *and* $t > T_0(\eta)$,

$$\sum_{j=1}^{n} m_{ij}^{\epsilon} > 1 - \eta, \quad i = 1, \ldots, N.$$

*Furthermore we have the following probability*

$$P(\mathbb{R}^n \setminus D) \leq \eta,$$

*where* $D$ *is the union of neighborhoods of stable critical points.*

*Proof.* Consider the Cauchy problem

$$\begin{cases} u_t &= \epsilon \Delta u + \nabla \cdot (u \nabla f), \quad x \in \mathbb{R}^n \\ u(0,x) &= u_0(x) \end{cases} \tag{4.10}$$

with $u_0$ satisfying $u_0 \geq 0$ and $\int_{\mathbb{R}^n} u_0 \mathrm{d}x = 1$. Note that (4.10) has a unique steady state $u^\epsilon(x) = \mathrm{e}^{-f(x)/\epsilon}$ and that any solution to (4.10) converges to the unique steady state when $t \to \infty$ and that the convergence is uniform on any compact sets in $\mathbb{R}^n$, see [21, 25] for details. Moreover, when $\epsilon \to 0$, the steady state $u^\epsilon$ converges to the Dirac measure concentrating on global minimum points of $f$, see [4, 17] for details.

Fix a small neighborhood $D$ satisfying $D \subset \cup_{i=1}^n B_i$, recalling that $B_i$ is the $\alpha$ neighborhood of critical point $x_i$ of $f$. For arbitrary $\eta > 0$, there exists an $\epsilon_0 > 0$ such that for any $\epsilon < \epsilon_0$,

$$\int_D u^\epsilon(x) \mathrm{d}x > 1 - \eta/2.$$

Then choose $T_0$ large enough such that, for given $\epsilon < \epsilon_0$,

$$\int_D |u(t,x) - u^\epsilon(x)| \mathrm{d}x < \eta/2, \quad \forall t \geq T_0.$$

Therefore, we have

$$\int_D u(t,x) \mathrm{d}x \geq \int_D u^\epsilon(x) \mathrm{d}x - \int_D |u^\epsilon(x) - u(t,x)| \mathrm{d}x \geq 1 - \eta/2 - \eta/2 \geq 1 - \eta.$$

By the definition of $m_{ij}^\epsilon$ and the choice of $D$, the result follows. □

## 5. Conley-Markov matrix for general systems

In this section, we extend the definitions of Conley-Markov matrix and natural order to general dissipative systems.

### 5.1. **A natural order for invariant sets.**

We consider the following general system perturbed by white noise

$$\mathrm{d}x = F(x)\mathrm{d}t + \sqrt{2\epsilon}\,\mathrm{d}W, \quad x \in \mathbb{R}^n.$$

The transition probability density function of the perturbed system satisfies the Fokker-Planck equation:

$$u_t = \epsilon \Delta u - \nabla \cdot (uF).$$

We assume that the unperturbed system

$$\mathrm{d}x = F(x)\mathrm{d}t \tag{5.1}$$

has a compact global attractor $\mathcal{A}$, which has a Morse decomposition $\{M_i\}_{i=1}^N$. Let us denote $\mathcal{N}_i$ as a small neighborhood of $M_i$, and also assume that they are disjoint.

Given a Morse decomposition, a very natural question to ask is: can we define some kind of ranking among the Morse sets? The answer would be simple if the system is a gradient like , for which the original energy can serve for the purpose. For a general system, an obvious total order can be defined by a Lyapunov function. Unfortunately, this total order is often non-unique. Different Lyapunov functions may produce different orders. On the other hand, there exist a very natural partial order '$<$' induced by the flow itself: $M_i < M_j$ if and only if there is some point $x$ such that the omega-limit of $x$ belongs to $M_i$ and the alpha-limit set of $x$ belongs to $M_j$, i.e. there is a connecting orbit from $M_j$ to $M_i$. It is clear that for any given Morse decomposition, the flow induced partial order is uniquely determined. In fact, the classical Conley connection matrix is an $N \times N$ matrix whose entries are maps between homology groups associated to the various Morse sets. These entries can be used to detect the the connecting orbits between different Morse sets, hence to determine the flow induced order between Morse sets. However, we need to compute the homology groups of each Morse set and the maps between these groups to get the connecting matrix, which may not come simple in general. More importantly, we must note that this flow induced order is only a partial order, because it is not possible to rank two Morse sets that have connecting orbit from another common Morse set.

We can also ask a more general question on whether there exist a total order for a finite number of disjoint invariant subsets of $S$?

Similar to the gradient systems, we will define Conley-Markov matrix for this Morse decomposition $\{M_i\}_{i=1}^N$ through the corresponding Fokker-Planck equation

$$u_t = \epsilon \Delta u - \nabla \cdot (uF), \quad u(0,x) = u_0^i(x), \quad x \in \mathbb{R}^n, \tag{5.2}$$

where $u^{\epsilon,i}(t,x) \geq 0$ is the solution satisfying,

$$\int_{\mathbb{R}^n} u^{\epsilon,i} \mathrm{d}x = 1.$$

We drop the super index $i$ in $u^{\epsilon,i}$ in the following discussion if no confusion is caused.

**Definition 5.1.** We can also define the average probability that the trajectories starting from $_i$ enters $M_j$ in the period $[0, T$ by

$$m_{ij}^{\epsilon} := \frac{1}{T}\int_0^T p_{ij}(t)\mathrm{d}t. \tag{5.3}$$

The *Conley-Markov matrix of the Morse decomposition* $\{M_i\}_{i=1}^N$ with $\epsilon$-diffusion, denoted by $M^\epsilon(T)$, of (5.1) is defined as

$$M^\epsilon(T) = \begin{pmatrix} m_{11}^\epsilon & m_{12}^\epsilon & \cdots & m_{1N}^\epsilon \\ m_{21}^\epsilon & m_{22}^\epsilon & \cdots & m_{2N}^\epsilon \\ \vdots & \vdots & \ddots & \vdots \\ m_{n1}^\epsilon & m_{n2}^\epsilon & \cdots & m_{NN}^\epsilon \end{pmatrix}.$$

Following the classical linear PDE theory, it is easy to verify that the Cauchy problem (5.2) admits a unique stationary solution $u^\epsilon$ satisfying

(1) $\epsilon\Delta u^\epsilon - \nabla\cdot(u^\epsilon F) = 0$ .
(2) For any Borel measurable set $B \subset \mathbb{R}^n$, we have $\lim_{t\to\infty}\int_B u(t,x)\mathrm{d}x = \int_B u^\epsilon(x)\mathrm{d}x$.

**Definition 5.2.** Let $S_j^\epsilon := \sum_{i=1}^N m_{ij}^\epsilon$, the sum of the $j$-th column of $M^\epsilon$, $j = 1,\dots,N$. $S_j$ is called the *natural energy of the Morse set* $M_j$.

**Definition 5.3.** The *natural order* '$\leqq_\epsilon$' between the Morse sets $\{M_j; j = 1,\dots,N\}$ is determined by

- $M_i <_\epsilon M_j$ if and only if $S_i^\epsilon < S_j^\epsilon$ for the path $\omega$.
- $M_i =_\epsilon M_j$ if $S_i^\epsilon = S_j^\epsilon$.

The natural energy $S_j^\epsilon$ denotes the total 'probability' that the trajectory starting from all the $\mathcal{N}_i$'s will enter $\mathcal{N}_j$ in the period $[0,T]$. Note that $S_j^\epsilon$ may be greater than 1 since $0 \le m_{ij} \le 1$ and hence $0 \le S_j^\epsilon \le N$.

If $S_k^\epsilon = \max_j S_j^\epsilon$, then $M_k$ is the most stable Morse set in probability sense. If we decompose the $M_k$ further and repeat the above argument, we can locate the most stable invariant set more precisely.

Thus, we use the Conley-Markov matrix to produce a natural order among disjoint invariant subsets of $S$. And we remark that the this method has the merits of being simple and practically useful in numerical simulations. More importantly, when there are several local minimal points for gradient systems ('minimal' invariant sets for general systems), the classical Conley index theory (Morse decomposition) can not distinguish which one is global minimal or most stable. By our Conley-Markov matrix, we can do this easily.

Next, we introduce a quantization strategy for the Conley-Markov matrix and compare it with the classical Conley connection matrix.

According to the decreasing natural order of the Morse sets in definition 5.3, we relabel the Morse sets and we still use $M_i$, $i = 1,\dots,N$ to denote the relabeled Morse sets. Then we obtain a relabeled Conley-Markov matrix with $\epsilon$-diffusion of the Morse decomposition $\{M_i\}_{i=1}^N$, which we still denote by $M^\epsilon$. For this rebeled matrix, we let

$$\tilde{m}_{ij} = \begin{cases} 0, & i \ge j, \\ 0, & i < j, m_{ij} = 0, \\ 1, & i < j, m_{ij} > 0. \end{cases} \tag{5.4}$$

Then we have the following definition.

**Definition 5.4.** *The matrix*

$$\tilde{M}^\epsilon = \begin{pmatrix} \tilde{m}_{11}^\epsilon & \tilde{m}_{12}^\epsilon & \cdots & \tilde{m}_{1N}^\epsilon \\ \tilde{m}_{21}^\epsilon & \tilde{m}_{22}^\epsilon & \cdots & \tilde{m}_{2N}^\epsilon \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{m}_{n1}^\epsilon & \tilde{m}_{n2}^\epsilon & \cdots & \tilde{m}_{NN}^\epsilon \end{pmatrix}$$

*with* $\tilde{m}_{ij}$ *given by* (5.4) *is called the* quantized connection matrix of the Morse decomposition $\{M_i\}_{i=1}^N$.

The quantized connection matrix can recover some information of the classical Conley connection matrix. For example, if $\tilde{m}_{ij} = 1$, then it indicates that there is a connecting orbit from the Morse set $M_i$ to $M_j$. More importantly, In the classical Conley connection matrix theory, if $m_{ij}$ is trivial, we can not conclude on whether there exists a connecting orbit from $M_i$ to $M_j$, for details, see [14]. On the other hand, we can detect all the connecting orbits by Conley-Markov matrix. In fact, we compute the probability from 'neighborhood' to 'neighborhood' instead of from invariant set to invariant set. This guarantees that if the connecting orbit, say from $M_i$ to $M_j$, exists, we can always obtain positive $m^\epsilon_{ij}$ for all sufficiently small $\epsilon > 0$ up to appropriate choices of the parameters. Hence in the quantized connection matrix, $\tilde{m}^\epsilon_{ij} = 1$, i.e. we can conclude the existence of connecting orbit.

5.2. **A generalized Morse decomposition.** We assume that $S$ is an invariant set, and $S_1, \dots, S_N$ are invariant subsets of $S$, if there exist disjoint neighborhoods $\mathcal{N}_1, \dots, \mathcal{N}_N$ of $S_1, \dots, S_n$. We also assume that nonnegative continuous functions $u_0^i$, $i = 1, \dots, N$, satisfy $\mathrm{supp}(u_0^i) \subset \mathrm{int}\mathcal{N}_i$ and $\int_{\mathbb{R}^n} u_0^i(x)\mathrm{d}x = 1$. Let us consider the Cauchy problem

$$\left\{ \begin{array}{rcl} u_t &=& \epsilon\Delta u - \nabla\cdot(uF), \quad x \in \mathbb{R}^n \\ u(0,x) &=& u_0^i(x) \end{array} \right. \tag{5.5}$$

and assume that $u^{\epsilon,i}(t,x)$ is the solution to (5.5). Then we can introduce a concept called generalized Morse decomposition.

**Definition 5.5.** Let

$$m^\epsilon_{ij} := \frac{1}{T}\int_0^T \int_{\mathcal{N}_j} u^{\epsilon,i}(t,x)\mathrm{d}x\mathrm{d}t, \quad 1 \le i, j \le N \tag{5.6}$$

and

$$M^\epsilon = \begin{pmatrix} m^\epsilon_{11} & m^\epsilon_{12} & \cdots & m^\epsilon_{1N} \\ m^\epsilon_{21} & m^\epsilon_{22} & \cdots & m^\epsilon_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ m^\epsilon_{n1} & m^\epsilon_{n2} & \cdots & m^\epsilon_{NN} \end{pmatrix},$$

where $u^{\epsilon,i}(t,x)$ is the solution to (5.5). If for arbitrary $\eta > 0$ and arbitrary disjoint neighborhoods $\mathcal{N}_1, \dots, \mathcal{N}_N$ of $S_1, \dots, S_n$, there exists $T_0 > 0$ and $\epsilon_0 > 0$ such that when $T \ge T_0$ and $\epsilon < \epsilon_0$,

$$\sum_j m_{ij} > 1 - \eta,$$

then $\{S_i; i = 1, \dots, N\}$ is called a *generalized Morse decomposition of $S$*.

It is clear that a Morse decomposition is a generalized one since the probability the trajectory lies on connecting orbits is small when $T$ is large and $\epsilon$ is small.

For the generalized Morse decomposition, we can also define the natural energy and the natural order.

**Definition 5.6.** Assume that $\{S_i; i = 1, \dots, N\}$ is a generalized Morse decomposition of $S$. Let $S^\epsilon_j := \sum_{i=1}^N m^\epsilon_{ij}$, the sum of the $j$-th column of $M^\epsilon$ in definition 5.5, $j = 1, \dots, N$. $S^\epsilon_j$ is called the *natural energy of the generalized Morse set $S_j$*.

**Definition 5.7.** The *natural order '$\leqq_\epsilon$'* between the generalized Morse sets $\{S_j; j = 1, \dots, N\}$ is determined by

- $S_i <_\epsilon S_j$ if and only if $S^\epsilon_i < S^\epsilon_j$ for all sufficiently small $\epsilon$;
- $S_i = S_j$ if $S^\epsilon_i \approx S^\epsilon_j$ for all sufficiently small $\epsilon$.

5.3. **Relation with deterministic case.** In previous subsections, to study the system

$$\mathrm{d}x = F(x)\mathrm{d}t, \quad x \in \mathbb{R}^n, \tag{5.7}$$

we consider the stochastically perturbed system

$$\mathrm{d}x = F(x)\mathrm{d}t + \sqrt{2\epsilon}\,\mathrm{d}W, \quad x \in \mathbb{R}^n.$$

A natural question is when the intensity of perturbation converges to zero, i.e. $\epsilon \to 0$, what can we conclude? In this subsection, we will discuss this issue.

Consider the Cauchy problem

$$\begin{cases} u_t + \nabla \cdot (uF) = 0, & x \in \mathbb{R}^n \\ u(0,x) = u_0(x) \end{cases} \tag{5.8}$$

where $u_0 \geq 0$ and $\int_{\mathbb{R}^n} u_0(x)\mathrm{d}x = 1$. Here for simplicity, we assume that $u_0$ is in $C^{2,\alpha}(\mathbb{R}^n)$, i.e. $\|u_0\|_{C^{2,\alpha}(\mathbb{R}^n)} < \infty$, where $C^{2,\alpha}(\mathbb{R}^n)$ is the space of twice continuously differentiable functions with the twice derivative being Hölder continuous with exponent $\alpha$.

Denote $D_{T,R} = [0,T] \times B_R(0)$, where $B_R(0)$ is the open ball centered at the origin with radius $R$. Let $C_0^1$ be the class of $C^1$ functions $\phi(t,x)$ which vanish outside of a compact subset in $t \geq 0$, i.e., $(\mathrm{supp}\phi) \cap (t \geq 0) \subset D_{T,R}$, and $\phi = 0$ when $t = T$ or $x \in \partial B_R(0)$. Following [27], we introduce the following definition.

**Definition 5.8.** A bounded measurable function $u(x,t)$ is called a *weak solution* of the Cauchy problem (5.8) with initial value $u_0$, provided that for all $\phi \in C_0^1$, the following holds

$$\iint_{\mathbb{R}^n \times [0,T]} (u\phi_t + uF \cdot \nabla\phi)\,\mathrm{d}x\mathrm{d}t + \int_{\mathbb{R}^n} u_0 \phi(0,x)\,\mathrm{d}x = 0. \tag{5.9}$$

**Theorem 4.** *Consider the Cauchy problem* (5.8). *Assume that* $|\mathrm{div}F|$ *is bounded in* $\mathbb{R}^n$. *Then there exists a unique weak solution* $u$ *to* (5.8) *which satisfies* $u = \lim_{\epsilon \to 0} u^\epsilon$ *in the* $L^1(\mathbb{R}^n \times [0,T])$ *topology, where* $u^\epsilon$ *is the solution to*

$$\begin{cases} u_t^\epsilon + \nabla \cdot (u^\epsilon F) = \epsilon \Delta u^\epsilon, & x \in \mathbb{R}^n \\ u^\epsilon(0,x) = u_0(x). \end{cases} \tag{5.10}$$

*In addition, there exist a matrix* $M^0(T)$ *such that,*

$$\lim_{\epsilon \to 0} M^\epsilon(T) = M^0(T). \tag{5.11}$$

*Proof.* We use vanishing viscosity method to prove the existence. Consider the viscous equation (5.10). It is known that (5.10) has a unique solution $u^\epsilon$ by classical parabolic theory, see for example [15, 18]. Furthermore, the following estimate for $u^\epsilon$ holds

$$\|u^\epsilon\|_{C^{2,\alpha}(\mathbb{R}^n \times (0,T))} \leq c\|u_0\|_{C^{2,\alpha}(\mathbb{R}^n)}, \tag{5.12}$$

for details see Theorem 6 on page 65 of [15] or (14.5) on page 390 of [18]. Actually, by writing the formula of the fundamental solution (dependent on $\epsilon$ of course) of (5.10) and noting the estimates for the fundamental solution (see (13.1) on page 376 of [18]), it is not hard to see that the constant $c$ in (5.12) is independent of $\epsilon$. Hence for any $R > 0$ and $T > 0$, we have

$$\iint_{D_{T,R}} |u^\epsilon|\mathrm{d}x\mathrm{d}t, \ \iint_{D_{T,R}} \left|\frac{\partial u^\epsilon}{\partial x_k}\right| \mathrm{d}x\mathrm{d}t, \ \iint_{D_{T,R}} \left|\frac{\partial u^\epsilon}{\partial t}\right| \mathrm{d}x\mathrm{d}t \leq M(R,T)$$

for $k = 1, \ldots, n$ and some constant $M(R,T)$ which is independent of $\epsilon$. These inequalities imply that the set $\{u^\epsilon\}$ is compact in the $L^1(D_{T,R})$ norm. By standard diagonal process, we can find a subsequence $\{u^{\epsilon_k}\}_{k=1}^\infty$ with $\epsilon^k \to 0$ converging almost everywhere in $\mathbb{R}^n \times [0,T]$ to a bounded function $u(t,x)$ when $k \to \infty$. We assert that this $u(t,x)$ is a solution to (5.8). Actually, let us consider the integration

$$\iint_{\mathbb{R}^n \times [0,T]} (u_t^\epsilon - \epsilon \Delta u^\epsilon + \nabla \cdot (u^\epsilon F))\phi \ \mathrm{d}x\mathrm{d}t$$

for arbitrary $\phi \in C_0^1$. Noting that $\phi$ has compact support, integrating by parts and by divergence theorem, we obtain

$$\begin{aligned}\iint_{\mathbb{R}^n \times [0,T]} u_t^\epsilon \phi \ \mathrm{d}x\mathrm{d}t &= \int_{\mathbb{R}^n} (u^\epsilon \phi)|_0^T \ \mathrm{d}x - \iint_{\mathbb{R}^n \times [0,T]} u^\epsilon \phi_t \ \mathrm{d}x\mathrm{d}t \\ &= -\int_{\mathbb{R}^n} u_0(x)\phi(x,0) \ \mathrm{d}x - \iint_{\mathbb{R}^n \times [0,T]} u^\epsilon \phi_t \ \mathrm{d}x\mathrm{d}t,\end{aligned}$$

$$\begin{aligned}\iint_{\mathbb{R}^n \times [0,T]} \Delta u^\epsilon \phi \ \mathrm{d}x\mathrm{d}t &= \iint_{B_R(0) \times [0,T]} \Delta u^\epsilon \phi \ \mathrm{d}x\mathrm{d}t \quad (\mathrm{supp}\phi \subset B_R(0) \times [0,T]) \\ &= \int_0^T \int_{\partial B_R(0)} \phi \nabla u^\epsilon \cdot \vec{n} \ \mathrm{d}s - \iint_{B_R(0) \times [0,T]} \nabla u^\epsilon \cdot \nabla \phi \ \mathrm{d}x\mathrm{d}t \\ &= -\iint_{\mathbb{R}^n \times [0,T]} \nabla u^\epsilon \cdot \nabla \phi \ \mathrm{d}x\mathrm{d}t,\end{aligned}$$

and similarly

$$\begin{aligned}\iint_{\mathbb{R}^n \times [0,T]} \nabla \cdot (u^\epsilon F)\phi \ \mathrm{d}x\mathrm{d}t &= \iint_{B_R(0) \times [0,T]} \nabla \cdot (u^\epsilon F)\phi \ \mathrm{d}x\mathrm{d}t \\ &= \int_0^T \int_{\partial B_R(0)} \phi u^\epsilon F \cdot \vec{n} \ \mathrm{d}s\mathrm{d}t - \iint_{B_R(0) \times [0,T]} u^\epsilon F \cdot \nabla \phi \ \mathrm{d}x\mathrm{d}t \\ &= -\iint_{\mathbb{R}^n \times [0,T]} u^\epsilon F \cdot \nabla \phi \ \mathrm{d}x\mathrm{d}t,\end{aligned}$$

where $\vec{n}$ stands for the unit outer normal of $\partial B_R(0)$ and $\mathrm{d}s$ stands for the $(n-1)$–dimensional area element in $\partial B_R(0)$. Noting that

$$\iint_{\mathbb{R}^n \times [0,T]} (u_t^\epsilon - \epsilon \Delta u^\epsilon + \nabla \cdot (u^\epsilon F))\phi \ \mathrm{d}x\mathrm{d}t = 0$$

since $u_t^\epsilon - \epsilon \Delta u^\epsilon + \nabla \cdot (u^\epsilon F) = 0$, we have

$$\begin{aligned}&\iint_{\mathbb{R}^n \times [0,T]} u^\epsilon \phi_t \ \mathrm{d}x\mathrm{d}t - \epsilon \iint_{\mathbb{R}^n \times [0,T]} \nabla u^\epsilon \cdot \nabla \phi \ \mathrm{d}x\mathrm{d}t \\ &+ \iint_{\mathbb{R}^n \times [0,T]} u^\epsilon F \cdot \nabla \phi \ \mathrm{d}x\mathrm{d}t + \int_{\mathbb{R}^n} u_0(x)\phi(x,0) \ \mathrm{d}x = 0.\end{aligned}$$

Letting $\epsilon \to 0$, we get

$$\iint_{\mathbb{R}^n \times [0,T]} (u\phi_t + uF \cdot \nabla \phi) \ \mathrm{d}x\mathrm{d}t + \int_{\mathbb{R}^n} u_0 \phi(x,0) \ \mathrm{d}x = 0.$$

Hence we obtain the existence of the solution.

Now let us consider the uniqueness. It suffices to prove that there is only zero solution to the Cauchy problem

$$\left\{ \begin{array}{l} u_t + \nabla \cdot (uF) = 0 \\ u(0,x) \equiv 0. \end{array} \right. \tag{5.13}$$

Multiplying $u$ and integrating with respect to $x$ in $\mathbb{R}^n$ on both sides of (5.13), we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2_{L^2(\mathbb{R}^n)} = -\int_{\mathbb{R}^n}\left(u(\nabla u\cdot F) + u^2\mathrm{div}F\right)\,\mathrm{d}x.$$

Since $|\mathrm{div}F|$ is bounded,

$$\int_{\mathbb{R}^n} u^2\mathrm{div}F\,\mathrm{d}x \le \text{const.}\|u(t)\|^2_{L^2(\mathbb{R}^n)}.$$

By integrating by parts, it follows that

$$\int_{\mathbb{R}^n} u(\nabla u\cdot F)\,\mathrm{d}x = -\frac{1}{2}\int_{\mathbb{R}^n} u^2\mathrm{div}F\,\mathrm{d}x,$$

so

$$\int_{\mathbb{R}^n} u(\nabla u\cdot F)\,\mathrm{d}x \le \text{const.}\|u(t)\|^2_{L^2(\mathbb{R}^n)}.$$

Therefore, we obtain that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2_{L^2(\mathbb{R}^n)} \le \text{const.}\|u\|^2_{L^2(\mathbb{R}^n)}.$$

Then Gronwall inequality enforces that $\|u(t)\|^2_{L^2(\mathbb{R}^n)} = 0$ for $t\in[0,T]$. That is, $u(t)=0$ almost everywhere for $t\in[0,T]$. The proof is complete. □

**Remark 5.1.** From the proof of theorem 4, it is not hard to check that if $x_0$ is a fixed point of the system $\dot{x}=F(x)$, i.e. $F(x_0)=0$, then the Dirac function $u(t,x)\equiv\delta_{x_0}$ is always a solution to (5.8) in the sense of (5.9). This coincides with our intuition. Actually, since the evolution of the density function for the transition probability of the system $\dot{x}=F(x)$ satisfies the non-viscous equation (5.8), then it is reasonable that this intuition should be true.

5.4. **An example for connection matrix.** We compute the connection matrix and the probability density function, the solution of Fokker-Planck equation $p(x,t)$, for the following 1-D equation,

$$dx = -\nabla f(x)dt + \epsilon dW_t, \tag{5.14}$$

where $f(x)$ is a potential function defined by

$$f(x) = -\alpha(x-1)^2 + x^4/10,$$

with $\alpha=5$ in the experiments, and $W_t$ is the standard Brownian motion.

As shown in Figure 1, there are three critical points for the corresponding deterministic equation, residing approximately at $x_1=-5.4$, $x_2=1$ and $x_3=4.4$.

Fokker-Planck equation for (5.14) is

$$(p_i)_t = \nabla(\nabla f(x)p_i) + \frac{1}{2}\epsilon\Delta p_i, \tag{5.15}$$

with initial condition as a Dirac $\delta$-function

$$p_i(x,t)|_{t=0} = \delta_{x_i}(x). \tag{5.16}$$

We use Crank-Nicholson scheme to solve the Fokker-Planck equation for $p(x,t)$ with $t\in[0,T]$, where $T=3$ in this example. We refer to [5] for the explicit formulas for numerical implementations.

Figures 3 - 6 shows the solutions $p$ at time $T$ for different parameter $\epsilon$ values as $6, 3, 1.5$ and $0.75$ respectively. In all plots, we use "blue" for the $p$ started at $x_1$, "green" for the $p$ started at $x_2$ and "red" for $x_3$. It is worth to point out that the probability density functions $p_2$ for the initial condition concentrated at $x_2=1$, the unstable critical point, become vanishing near $x_2$ regardless the value of $\epsilon$, while $p_1$ and $p_3$ still clustered near their critical points respectively, especially when $\epsilon$ becomes smaller.

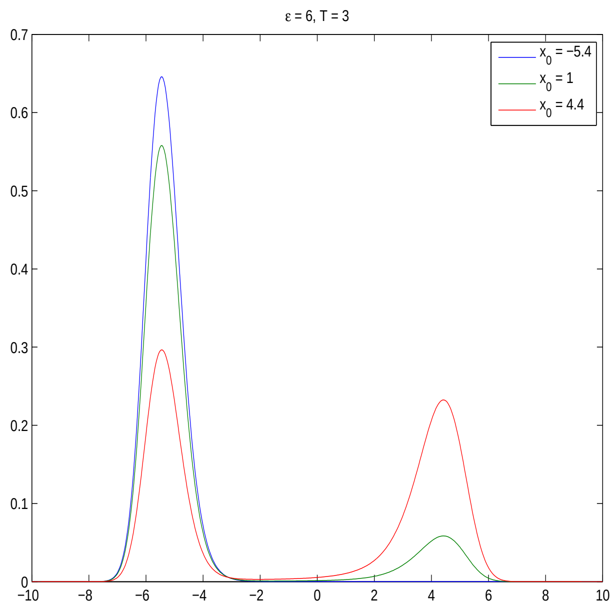


FIGURE 3. The probability density functions with three different initial conditions at $x_i$ and time $T = 3$ and parameter $\epsilon = 6$.

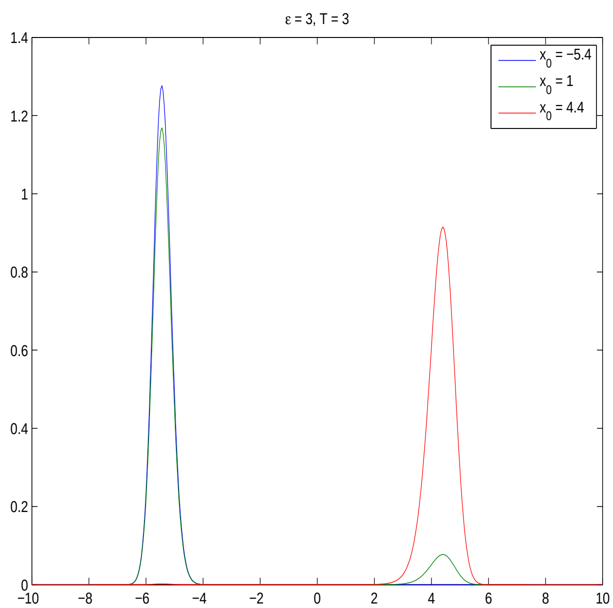


FIGURE 4. The probability density functions with three different initial conditions at $x_i$ and time $T = 3$ and parameter $\epsilon = 3$.

The time dependent connection matrix (the definition maybe different from what you gave in the draft) is defined as

$$C(t) = [c_{ij}(t)], \tag{5.17}$$

where $m_{ij}(t)$ is defined by

$$c_{ij}(t) = \int_{B_\delta(x_j)} p_i(x,t)dx, \tag{5.18}$$

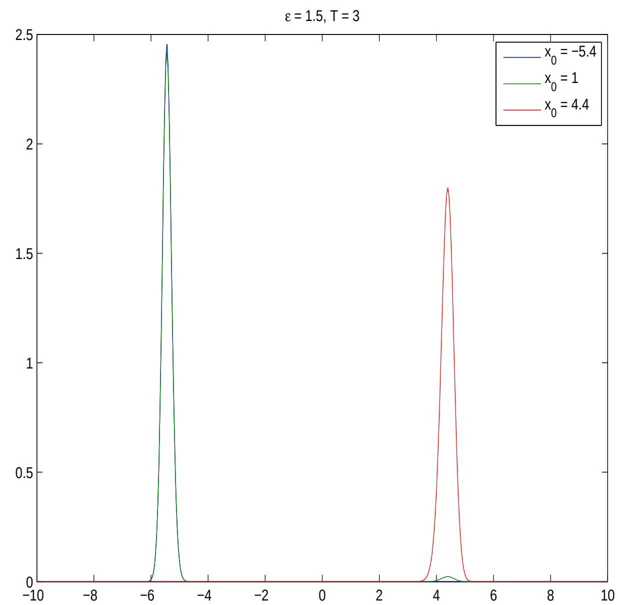


FIGURE 5. The probability density functions with three different initial conditions at $x_i$ and time $T = 3$ and parameter $\epsilon = 1.5$.

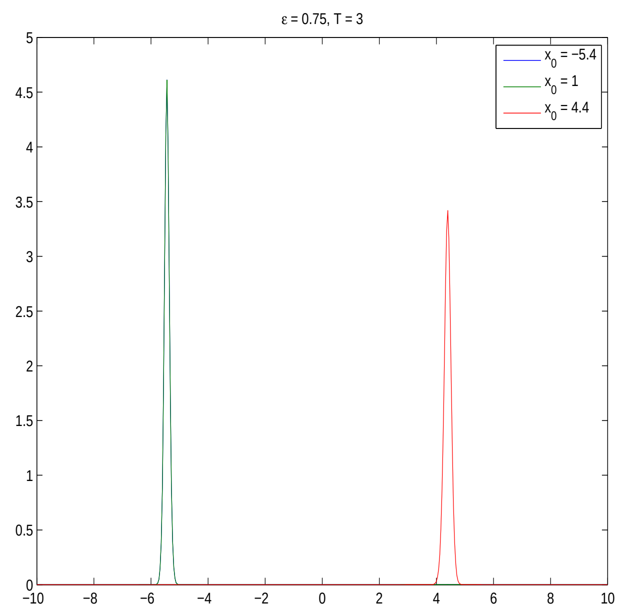


FIGURE 6. The probability density functions with three different initial conditions at $x_i$ and time $T = 3$ and parameter $\epsilon = 0.75$.

and $B_\delta(x_j)$ is a small ball centered at $x_j$ with radius $\delta$. Clearly, $c_{ij}(t)$ is the probability that the trajectories originated from $x_i$ reach a small neighborhood of $x_j$ at time $t$.

To consider the total probability of the trajectories started with $x_i$ and at least reach a small ball around $x_j$ during the period $[0, T]$, we defined a finite time connection matrix

$$M = [m_{ij}], \tag{5.19}$$

| $\epsilon$ | $M$ | $C(t)$ |
|---|---|---|
| 6 | $\begin{bmatrix} 0.4829 & 0.0000 & 0.0000 \\ 0.3133 & 0.0188 & 0.0870 \\ 0.0913 & 0.0106 & 0.2753 \end{bmatrix}$ | $\begin{bmatrix} 0.8139 & 0.0000 & 0.0000 \\ 0.7452 & 0.0000 & 0.0556 \\ 0.0012 & 0.0001 & 0.6587 \end{bmatrix}$ |
| 3 | $\begin{bmatrix} 0.7931 & 0.0000 & 0.0000 \\ 0.5315 & 0.0263 & 0.1282 \\ 0.0003 & 0.0000 & 0.6412 \end{bmatrix}$ | $\begin{bmatrix} 0.8139 & 0.0000 & 0.0000 \\ 0.7452 & 0.0000 & 0.0556 \\ 0.0012 & 0.0001 & 0.6587 \end{bmatrix}$ |
| 1.5 | $\begin{bmatrix} 0.9791 & 0.0000 & 0.0000 \\ 0.7245 & 0.0420 & 0.0959 \\ 0.0000 & 0.0000 & 0.9213 \end{bmatrix}$ | $\begin{bmatrix} 0.9849 & 0.0000 & 0.0000 \\ 0.9722 & 0.0000 & 0.0120 \\ 0.0000 & 0.0000 & 0.9336 \end{bmatrix}$ |
| 0.75 | $\begin{bmatrix} 0.9998 & 0.0000 & 0.0000 \\ 0.7853 & 0.0597 & 0.0406 \\ 0.0000 & 0.0000 & 0.9980 \end{bmatrix}$ | $\begin{bmatrix} 0.9999 & 0.0000 & 0.0000 \\ 0.9996 & 0.0000 & 0.0003 \\ 0.0000 & 0.0000 & 0.9988 \end{bmatrix}$ |

TABLE 1. The connection matrices for different parameter $\epsilon$ values.

where

$$m_{ij} = \frac{1}{T}\int_0^T c_{ij}(t)dt. \tag{5.20}$$

The difference between $C(t)$ and $M$ is that $C(t)$ describe an instant property at time $t$ while $M$ is the accumulated information in the period $[0, T]$. The trajectories contributed to $M$ may not stay within the small ball $B_{delta}(x_j)$ through out the period.

In Table 1, we present the two matrices $M$ and $C(t)$ with $t = T = 3$, and parameter $\epsilon = 6, 3, 1.5$, and $0.75$ respectively.

We note that for all the matrices (both $C(t)$ and $M$), the first column always has the largest sum, the second column always has the smallest sum, and the third column is in the middle. This gives an order $(x_1, x_3, x_2)$ to the corresponding critical points that $x_1$ is the global minimizer. We also note that as $\epsilon$ becomes smaller, $c_{11}$ and $c_{33}$ are close to 1, which can be interpreted as that both of the points are essentially stable points. The probability of staying at the corresponding point becomes 1 when the intensity of noise diminished. On the other hand, $c_{22}$ remains close to zero, and both $c_{12}$ and $c_{32}$ are no-negative regardless how small $\epsilon$ is. In this sense, $x_2$ is a critical point that is nearly invisible if noise (does not matter how small it is) is present, so it can be excluded from the essential sets of critical points.

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